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Stein covers for curved twistor spaces

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Abstract. *We show that any curved twistor space has a naturally-defined Stein cover, the elements of which are indexed by the points of the twistor space. We use this cover to give compact formulae for the Penrose transform and the inverse twistor functions, and to provide a broader and less singular definition of googly twistor spaces than previously available.*

1. INTRODUCTION

A space-time is a four-dimensional manifold with a Lorentzian metric. The metric satisfies the Einstein vacuum equations if its Ricci curvature vanishes. In this case, the Bianchi identities satisfied by the Riemann tensor, when written in spinor notation, formally resemble the linear equations satisfied by a massless field of helicity two.

Motivated by this and a hope of interpreting the curvature spinor field in quantum theory, Penrose considered the condition for the field to be purely circularly polarized, which turns out to be the requirement that the Riemann tensor be (anti-) self-dual. Such a space-time he called a *non-linear graviton.* In

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fact, the algebraic requirement of (anti-) self-duality implies, for a Lorentzian metric, that the curvature tensor is complex, and thus non-linear gravitons are complex manifolds with metrics which are non-degenerate symmetric holomorphic forms on the holomorphic vectors.

With some mild provisos, there is a one-to-one correspondence between non-linear gravitons and *curved twistor spaces*, which are three-complex dimensional manifolds which fibre over the Riemann sphere (and have some additional structures) [14]. Thus space-times with (anti-) self-dual curvature can be studied by means of their twistor spaces, and it turns out that the local metric geometry of the space-time is coded in the global analytic geometry of the twistor spaces. One therefore wishes, in order to describe the space-time, to have tools with which to study the holomorphic geometry of the twistor space.

In this paper, we show that the twistor space has a natural Stein cover. The elements of this cover are indexed by the points in the twistor space, so the cover has *itself* the structure of a complex manifold. We give two applications.

The first is to the Penrose transform, which establishes isomorphisms between analytic first cohomology groups on twistor space and spaces of massless fields on space-time [3]. If the cohomology classes are represented by Čech cocycles with respect to the cover, these isomorphisms and their inverses are given by simple integral formulae. Versions of these formulae have been known for some time – indeed, they preceeded the cohomological results – but without the proper choice of cover many of them are too awkward to use for any but local (in space-time) computations [4, 13].

The second application is to the theory of googly twistor spaces. Twistors have a sort of handedness, and properly speaking one distinguishes between dual twistors and twistors. The non-linear graviton-curved twistor space, or *leg-break*, correspondence links space-times with anti-self-dual curvature to twistor spaces, and space-times with self-dual curvature to dual twistor spaces. The *googly problem* is to describe anti-self-dual curvatures with dual twistor spaces and self-dual curvatures with twistor spaces. The solution of this problem would hopefully bring us a step closer to the twistor description of space-times with non-self-dual curvatures. This is one of the main goals of twistor theory.

The most popular current program requires that the googly twistor space be constructed from the asymptotic structure of the space-time [10, 15]. Despite some suggestive partial results, this project has been frustrated by the lack of a good enough understanding of what these asymptotics should be. Too, naively at least, it seems odd that one should have to consider the global properties of space-time for the googly, when the leg-break can be accomplished locally. The Stein cover introduced here will be used to provide a *local* definition of googly twistor space (actually, a family of googly twistor spaces). It is also possible to use the cover to give the *googly maps*, which give the googly description of space-time points. The definition we give is applicable to a broader class of space-times than previous ones. The techniques introduced here are closely related to (and generalize) those figuring in other authors' recent work [11, 17, 19].

The organization of this paper is as follows. The next section reviews the non-linear graviton-curved twistor space correspondance, the section after introduces the Stein cover, and the last two sections cover the applications mentioned above.

For general background on twistors, see [3, 8, 9, 13, 20, 21] and for complex manifolds [7, 12]. For the googly, see [5, 10, 15 - 19].

Notation and conventions. Most of the assumptions and notation peculiar to this paper are introduced in section 2. Generally, we follow Penrose and Rindler [21]. In particular, the abstract index convention is used throughout, so vectors, spinors, etc. are represented by symbols with indices which do not take numeric values, but indicate to which space the object belongs.

If V is a vector space or vector bundle, PV denotes the associated projective space or bundle. The fibre of a bundle E over b is denoted E_b; the bundle restricted to a subset U of the base is denoted $E|_U$.

We work in the category of complex manifolds and holomorphic maps; all structures considered will be complex and holomorphic, and we omit saying so explicitly.

2. THE NON-LINEAR GRAVITON

We review here the non-linear graviton [14]. Let $(\mathscr{M}, g_{ab})$ be a complex space-time with anti-self-dual Riemann tensor. We denote the spin bundles $\mathscr{S}^A$, $\mathscr{S}^{A'}$. (We assume the space-time has a spin structure).

The anti-self-duality of R_{abcd} implies that there is a local parallelims of primed spinors, i.e. $\nabla_{[a}\nabla_{b]}\lambda_{C'} = 0$ for any $\lambda_{A'}$. We assume

(a) The local parallelism of primed spinors extends to a global parallelism. Thus $\mathscr{S}^{A'} \simeq \mathscr{M} \times \mathbb{S}^{A'}$ canonically where $\mathbb{S}^{A'} \simeq \mathbb{C}^2$ is *primed spin space*.

A second consequence of the anti-self-duality is that *self*-dual two-plane elements are locally integrable to two-surfaces. We suppose

(b) The local integrability of self-dual two-plane elements extends to a global integrability, and the space of these two-surfaces is in fact a manifold.

Each self-dual two-surface Z is called an α-surface; it is totally null and any tangent vector is of the form $\lambda^A \pi^{A'}$ for some fixed covariantly constant spinor $\pi^{A'}$, called the *tangent spinor* of Z. The space of α-surfaces is a three-dimensional manifold $\mathscr{P}$, *twistor space*. When we think of Z as a point in $\mathscr{P}$ we call it a *twistor*. Note that the «tangent spinor» of Z is really a projective spinor. The map $\mathscr{P} \to P\,\mathbb{S}_{A'}$ given by $Z \to \pi_{A'}$ as above will be denoted π.

Each $x \in \mathscr{M}$ determines a projective line

$$L_x = \{Z \in \mathscr{P} \mid x \in Z\}$$

embedded in $\mathscr{P}$. (There is one α-surface through x for each tangent projective spinor at x). $x, y \in \mathscr{M}$ are null-separated iff they lie on a common α-surface iff L_x meets L_y. We assume

(c) $L_x \cap L_y$ is empty or a singleton if $x \neq y$;

this is analogous to requiring that there are no conjugate points. Then if Q, $R \in \mathscr{P}$ with $Q \neq R$, there is at most one $x \in \mathscr{M}$ with $Q, R \in L_x$.

Notation. For x, Q, R as above, we put $x_{QR} = x$. We write $x_{QR} \in \mathscr{M}$ to mean there is $x \in \mathscr{M}$ with $Q, R \in L_x$. $Z(x, \pi_{A'})$ is the α-surface through x with tangent spinor $\pi_{A'}$.

Our last assumptions are

(d) $\mathscr{M}$ is Stein, and

(e) Each α-surface is connected and simply-connected.

Remark. (a) - (e) hold locally, i.e. if $x \in \mathscr{M}$, there are always neighborhoods $\mathscr{M}'$ of x for which (a) - (e) hold on $(\mathscr{M}', g_{ab}|_{\mathscr{M}'})$.

There is a line bundle $\mathscr{T}$ over $\mathscr{P}$, whose fibre over Z is $\{\pi_{A'} \in S_{A'} \mid$ taken projectively, $\pi_{A'}$ is the tangent spinor to $Z\}$. $\mathscr{T}$ is called *non-projective twistor space*. A non-projective twistor is thus an α-surface together with a choice of scale for its tangent spinor. The sheaf of germs of functions with values in $\mathscr{T}^n$, or equivalently homogeneous of degree $-n$ in $\pi_{A'}$, is denoted $\mathcal{O}(-n)$.

The results on massless fields are as follows. Let $\mathscr{Z}(s)$ be the space of massless fields of helicity s on $\mathscr{M}$. Then, under (a) - (e),

THEOREM (Eastwood, Penrose and Wells [3]). *For* $s \geqslant -1$,

$$\mathscr{Z}(s) = H^1(\mathscr{P}, \mathcal{O}(-2-2s)). \qquad \blacksquare$$

In general, there are no massless fields of helicity < -1 [1, 2, 21]. A *Maxwell field* on $\mathscr{M}$ is a connection on a line bundle over $\mathscr{M}$. The field is *anti-self-dual* if its curvature is.

THEOREM (Ward [23]; Eastwood, Penrose and Wells [3]). *The space of anti-self-dual Maxwell fields on* $\mathscr{M}$ *is isomorphic to* $H^1(\mathscr{P}, \mathcal{O}^*)$. *If the curvature of the field is regarded as a helicity* -1 *massless field* $\phi_{AB} \in \mathscr{Z}(s)$, *the cohomology element is determined as the image of the exponential map*

$$H^1(\mathscr{P}, \mathcal{O}) \to H^1(\mathscr{P}, \mathcal{O}^*). \qquad \blacksquare$$

3. THE COVER

In this section, we describe the Stein cover of $\mathscr{P}$.

DEFINITION 3.1. *For each $A \in \mathscr{P}$, let*

$$U_A = \{Z \in \mathscr{P} - A \mid x_{AZ} \in \mathscr{M}\}.$$

Let $\mathscr{U} = \{U_A \mid A \in \mathscr{P}\}$. ■

LEMMA 3.2. *Let $A \in \mathscr{P}$, and $\mathscr{B}$ be the primed projective tangent spin bundle restricted to A minus the tangent section. Then U_A is biholomorphic to $\mathscr{B}$.*

Proof. The biholomorphism is

$$Z \to (x_{AZ}, \pi(Z)).$$

It is well-defined by condition (b). The inverse is

$$(x, \pi_{A'}) \to Z(x, \pi_{A'}).$$ ■

THEOREM 3.3. *$\mathscr{U}$ is a Stein cover of $\mathscr{P}$.*

Proof. First, we show U_A is Stein. By the previous proposition, it is biholomorphic to $\mathscr{B}$. Now, $\mathscr{B}$ is biholomorphic to the product $(P\mathbb{S}_{A'} - \{\pi_{A'}\}) \times A$, where $\pi_{A'}$ is the tangent spinor to A, by (a) of section 2, i.e. to $\mathbb{C} \times A$. A is a closed submanifold of the Stein manifold $\mathscr{M}$, hence A is Stein. Thus $\mathbb{C} \times A$ is Stein, and U_A is.

Now, we show that $\mathscr{U}$ covers $\mathscr{P}$. For any $Z \in \mathscr{M}$, pick $x \in \mathscr{P}$ on the α-surface Z, and $A \in L_x$ with $A \neq Z$. Then $Z \in U_A$. ■

Remark. If one only assumes the Weyl tensor of $(\mathscr{M}, g_{ab})$ is anti-self-dual, the self-dual two-plane elements are still locally integrable and so a twistor space $\mathscr{P}$ exists. Replacing $\mathscr{M}$ by an open submanifold if necessary, this space is a three-dimensional manifold with all the structures assumed above except the fibration $\mathscr{P} \to P\mathbb{S}_{A'}$ (indeed, since when Ricci curvature is present there will not be even a local parallelism of primed spinors, $P\mathbb{S}_{A'}$ does not exist). One can still form the cover $\mathscr{U}$ and it is Stein if the second Betti number of each A vanishes, although the argument needed to prove this is more sophisticated. In outline, it is this: U_A is biholomorphic to $\mathscr{B}$ as before, but $\mathscr{B}$ has no obvious product structure. Instead, $\mathscr{B}$ is a $\mathbb{C}$-bundle over A with structure group G the affine group of $\mathbb{C}$. Thus $\mathscr{B}$ is classified by an element of $H^1(A, \mathscr{G})$, where $\mathscr{G}$ is the sheaf of germs of holomorphic functions with values in G. G is an extension of $\mathbb{C}$ by $\mathbb{C}^*$, and a little diagram-chasing shows $H^1(A, \mathscr{G}) = H^1(A, \mathcal{O}^*) = H^2(A, \mathbb{Z})$.

4. MASSLESS FIELDS

In this section, we treat the representation of analytic first cohomology elements by Čech cocycles relative to $\mathscr{U}$. Such cohomology elements are known to correspond to massless fields on $(\mathscr{M}, g_{ab})$:

THEOREM (Eastwood, Penrose and Wells). *There is, for $s \geqslant -1$, an isomorphism*

$$H^1(\mathscr{P}, \mathscr{O}(-2-2s)) \to \mathscr{Z}(s).$$

This map is essentially restriction to the line L_x at which the field is to be evaluated, followed by the isomorphism of Serre duality [22] (if $s = -\frac{1}{2}$ or -1, the restriction is actually to a formal neighborhood of L_x). It can thus be represented by a simple contour integral over L_x of any Čech representative.

The inverse map has been more awkward to describe. A Čech representative for the cohomology element corresponding to a given space-time field is called an *inverse twistor function*. Formulae for inverse twistor functions have been given «locally», for neighborhoods of L_x of the form $U_1 \cup U_2$, but this restriction has limited their usefulness. We show here that the cover $\mathscr{U}$ is naturally adapted to these formulae.

As corollaries of these results, we show that the elements of the cocycles relative to $\mathscr{U}$ can be chosen to vary holomorphically with the elements of $\mathscr{U}$. In the case $s = -1$, the massless field is an anti-self-dual electromagnetic field, and the holomorphically-varying cocycles encode the integral geometry of the gauge field. For simplicity, we shall assume $\mathscr{M}$ is simply-connected.

Let $\phi_{AB} \in \mathscr{Z}(-1)$. The assumption that $\mathscr{M}$ is simply-connected implies the existence of a potential Φ_a such that

$$\nabla_{A(A'}\Phi^{A}{}_{B')} = 0, \qquad \nabla_{A'(A}\Phi^{A'}{}_{B)} = \phi_{AB}.$$

Φ_a is determined only up to the addition of a gradient; we shall suppose some definite choice to have been made. It may be interpreted as a connection one-form on a line bundle over $\mathscr{M}$. Then $\phi_{AB}\epsilon_{A'B'}$ is the curvature of the connection. The freedom in choosing Φ_a is exactly the freedom in fibre coordinatization of the line bundle. A similar construction applies to Yang-Mills fields.

Define a Čech 1-cochain with respect to $\mathscr{U}$ by $f = \{f_{AB} \mid A, B \in \mathscr{P}\}$,

$$f_{AB}(Z) = \int_{x_{AZ}}^{x_{BZ}} \Phi_a \, dx^a \tag{1}$$

where the path of integration lies along the α-surface Z. The homotopy class of this path is well-defined for $Z \in U_A \cap U_B$, for then x_{AZ}, x_{BZ} are points on the α-surface Z, and this is connected and simply-connected. The integral is unchanged by deformations of the path, since

$$d(\Phi_{AA'}\,\mathrm{d}x^{AA'})|_{Z\text{ fixed}} = \nabla_{AA'}\Phi_{BB'}\,\mathrm{d}x^{A(A'}\wedge \mathrm{d}x^{B')B}$$
$$= \frac{1}{2}\epsilon_{AB}\nabla_{R(A'}\Phi^{R}{}_{B')}\,\mathrm{d}x^{AA'}\wedge \mathrm{d}x^{BB'} = 0.$$

(The surface element of Z is proportional to $\pi_{A'}\pi_{B'}$, hence only the term symmetric in A', B' survives). Since x_{AB}, x_{BZ} very holomorphically with Z, $f_{AB}(Z)$ is holomorphic. Further

$$f_{BA}(Z) = \int_{x_{BZ}}^{x_{AZ}} \Phi_a\,\mathrm{d}x^a = -f_{BA}(Z), \tag{2}$$

$$f_{AB}(Z) + f_{BC}(Z) = \left(\int_{x_{AZ}}^{x_{BZ}} + \int_{x_{BZ}}^{x_{CZ}}\right)\Phi_a\,\mathrm{d}x^a = \int_{x_{AZ}}^{x_{CZ}} \Phi_a\,\mathrm{d}x^a = f_{AC}(Z).$$

Thus f is a 1-cocycle.

We now turn to the interpretation of Φ_a as a gauge field. Denote by E the line bundle over $\mathscr{M}$ on which it is a connection. The anti-self-duality of the curvature $\phi_{AB}\epsilon_{A'B'}$ implies that it vanishes on each α-surface Z. Since Z is connected and simply-connected, the gauge field is trivial on Z. Thus we may form a line bundle $\mathscr{E}$ over $\mathscr{P}$, whose fibre over Z is the space of fields on Z with values in E which are gauge-covariantly constant on Z. We choose a fibre coordinate ζ_A on $\mathscr{E}|_{U_A}$ as follows. For $\zeta \in \mathscr{E}_Z$, let

$$\zeta_A(Z) = \text{value of the field } \zeta \text{ at the point } x_{AZ}.$$

Then the transition function from ζ_A to ζ_B over $U_A \cap U_B$ is $\exp f_{AB}(Z)$, since this is the parallel-propagation operator from x_{AZ} to x_{BZ}. The usual analysis of the Ward construction then shows ϕ_{AB} is the field derived from f by the isomorphism of Eastwood, Penrose and Wells.

To summarize, we have the following:

THEOREM 4.1. *For any cohomology element $f \in H^1(\mathscr{P}, \mathcal{O})$, there exists a cocycle $f = \{f_{AB}\}$ representing f which varies holomorphically with A, B. For the line bundle $\mathscr{E}$ on $\mathscr{P}$ representing the gauge field, there is a fibre coordinate ζ_A over U_A varying holomorphically with A for which the transition functions are* $\{\exp f_{AB}\}$. ■

For higher helicities, one has similar results.

THEOREM 4.2. *Let $f \in H^1(\mathscr{P}, \mathcal{O}(-2-2s))$ where $s \geqslant -\frac{1}{2}$. Then there is a cocycle $f = \{f_{AB}\}$ representing f and varying holomorphically with A, B.*

Proof. We shall give explicit formulae for the cocycles. Let the space-time field corresponding to f be ν_A (if $s = -\frac{1}{2}$) or $\phi_{A' \ldots C'}$ ($2s$ indices, if $s > -\frac{1}{2}$). Define a cocycle $f = \{f_{AB}\}$ by

$$f_{AB} = \int_{x_{AZ}}^{x_{BZ}} (\pi \cdot \xi)^{-1} \nu_C \xi_{C'} \, \mathrm{d}x^{CC'} \quad \text{if} \quad s = -\frac{1}{2}; \tag{3a}$$

$$= \int_{(x_{AZ}, \pi(A))}^{(x_{BZ}, \pi(B))} (\pi \cdot \xi)^{-2-2s} (\xi^{C'} \xi_{B'} \nabla_{CC'} \phi \, \mathrm{d}x^{CB'} + (2s+1)\phi \xi_{A'} \, \mathrm{d}\xi^{A'}) \tag{3b}$$

where $\phi = \phi_{A' \ldots C'} \xi^{A'} \ldots \xi^{C'}$ if $s > -\frac{1}{2}$.

The path of integration in (3a) is the same as that in (1) above. For (3b), it is any lift of that path to $P\mathscr{S}^{A'}$ such that $\xi \cdot \pi \neq 0$, $\xi_{A'} = \pi(A)$ at the initial point and $\xi_{A'} = \pi(B)$ at the final point. We remark (a) In the first formula, since $\mathrm{d}x^{AA'}$ is proportional to $\pi_{A'}$, the integrand is in fact independent of $\xi^{A'}$, (b) If the second path is regarded as a path in $\mathscr{P}$, it runs from A to B, (c) Since the spin bundle (minus its tangent section) is topologically a trivial bundle over Z, there is, up to homotopy, only one path for the second integral, (d) A straightforward calculation shows that the integrands are closed forms; we omit this, (e) f satisfies the cochain and cocycle conditions as above (equation 2).

We must now show that the cocycles defined here are indeed the ones corresponding to the space-time fields in the integrands under the isomorphism of [3]. Consider first the case $s > -\frac{1}{2}$. Then the isomorphism is defined by Serre duality as

$$f \to f|_{L_x} \Delta\pi \in H^1(L_x, \mathcal{O}(-2-2s) \otimes \Omega) = (H^0(L_x, \mathcal{O}(2s))^* = \$_{A' \ldots C'}$$

where $\Delta\pi = \pi_{A'} \, \mathrm{d}\pi^{A'}$ and Ω is the sheaf of germs of holomorphic 1-forms on L_x.

It will thus sufficient to compute $f|_{L_x}$.

Note that $\mathscr{U}_x = \{U_A \cap L_x\}$ is a refinement of $\mathscr{V}_x = \{L_x - \{A\} \mid A \in L_x\}$, the element $U_A \cap L_x$ being included in $L_x - \{B\}$ where $\pi(A) = \pi(B)$. Since also $\mathscr{V}_x \subset \mathscr{U}_x$, we may compute the cohomology of L_x using the cover $\mathscr{V}_x$. If $A, B \in L_x$ with $\pi(A) = \alpha_{A'}$, $\pi(B) = \beta_{A'}$, then

$$f_{AB} = (2s+1)\int_{\alpha_{A'}}^{\beta_{A'}} (\pi\cdot\xi)^{-2-2s}\phi(x)\Delta\xi$$

$$= (-1)^{2s}(2s!)^{-1}\phi_{A'\dots C'}\frac{\partial}{\partial\pi_{A'}}\cdots\frac{\partial}{\partial\pi_{C'}}\int_{\alpha_{A'}}^{\beta_{A'}}(\pi\cdot\xi)^{-2}\Delta\xi$$

$$= (-1)^{2s}(2s!)^{-1}\phi_{A'\dots C'}\frac{\partial}{\partial\pi_{A'}}\cdots\frac{\partial}{\partial\pi_{C'}}\frac{\alpha\cdot\beta}{\alpha\cdot\pi\beta\cdot\pi}.$$

The field evaluated from this cohomology class is

$$(2\pi i)^{-1}\int f_{AB}\pi_{A'}\dots\pi_{C'}\Delta\pi$$

where there are $2s$ π's and the contour separates A and B and surrounds A in the positive sense. Integrating by parts, we get

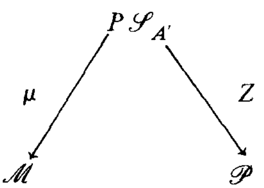
$$\phi_{A'\dots C'}(x).$$

Thus f does indeed represent the cohomology element corresponding to $\phi_{A'\dots C'}(x)$. Clearly, it varies holomorphically with A and B.

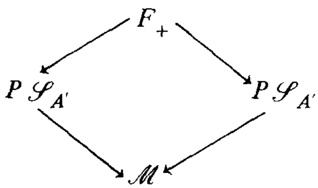
For $s = -\frac{1}{2}$, a similar argument applies, but one works on the first formal neighborhood of L_x. We omit the details, which are outlined in [6]. ∎

Remark. For $s \geqslant -\frac{1}{2}$, our formulae give preferred Čech representatives for the cohomology classes. It is not hard to see that these are characterized as those for which $f_{AB}(Z)$ is independent of Z when $Z \in L_{x_{AB}}$ (where defined). For $s = 0$, we get a preferred representative for any choice of fibre coordinatization.

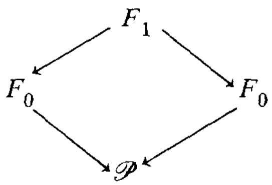
These constructions lead one to consider a particular series of manifolds. Note that there is a double fibration



where μ is the defining projection and Z is as in section 2. Form the pull-back



and let $F_0 = F_+$-the diagonal. Then a 0-cochain $f = \{f_A\}$ which varies holomorphically with A is simply a function on F_0. Define F_1 as the pull-back



A 1-cochain $f = \{f_{AB}\}$ which varies holomorphically with A, B is an antisymmetric function on F_1. Similarly, one can form $F_n = F_{n-1} \times_{\mathcal{P}} F_0$, and study the cohomology of the complex $\mathcal{O}(F_\bullet)$ (with the obvious coboundary maps). This has something of the flavor of Alexander-Spanier cohomology. We shall not pursue this here, and only remark that an analog of this complex can be used to compute the *googly cohomology groups* introduced in [5].

5. GOOGLY TWISTOR SPACE

The googly problem is to specify a googly twistor space $\mathcal{P}^*$ (of twistors with the opposite handedness to those in $\mathcal{P}$) and data on it from which one can recover a space-time with anti-self-dual curvature. As a first step to solving this, one turns the question around and asks how to define $\mathcal{P}^*$ and data on it from an anti-self-dual $(\mathcal{M}, g_{ab})$. The data we are concerned with here are the «googly maps», which represent the points of $\mathcal{M}$.

In outline, the construction of $\mathcal{P}^*$ and the googly maps in this [15]. Require $(\mathcal{M}, g_{ab})$ to be asymptotically flat in a suitable sense, so that a portion of complex infinity $\mathcal{N}$ exists. The asymptotics of $(\mathcal{M}, g_{ab})$ must be such that $\mathcal{N}$ is a subspace of the light-cone of a point $i \in \mathcal{N}$. Then $\mathcal{I} = \mathcal{N} - \{i\}$ is biholomorphic to a subset of $\mathbb{C}P_1 \times \mathbb{C}P_1 \times \mathbb{C} = \{(\pi_{A'}, \eta_A, u)\}$. The projective spinors here are taken relative to i and if a null ray tends to a point $(\pi_{A'}, \eta_A, u) \in \mathcal{I}$, $\pi_{A'}$ η_A determines its asymptotic direction and u the «retarded time» at which it meets $\mathcal{I}$. The anti-self-duality of the curvature then guarantees the existence of a three-dimensional family of null geodesics in η_A = constant surfaces of $\mathcal{I}$ which in flat space-time would be identified as the restriction of the *anti*-self-dual two-surfaces to $\mathcal{I}$. This space is (*asymptotic*) *dual twistor space* $\mathcal{P}^*$. When we wish to emphasize that we are thinking of a dual twistor as a locus on $\mathcal{I}$, we call it a dual twistor *line*. Any

$x \in \mathscr{M}$ determines a *googly map* $\gamma_x : \mathscr{P}^* \to PS_{A'}$ as follows. Let C_x be the intersection of the light-cone of x with $\mathscr{I}$. Then any dual twistor line W meets C_x in a point $(\pi_{A'}, \eta_A, u)$. We set $\gamma_x(W) = \pi_{A'}$. The googly maps are supposed to allow one to recover the space-time from $\mathscr{P}^*$.

There are a number of difficulties, long recognized, with this construction. One is the lack of a satisfactory understanding of what a complex asymptotically flat space-time should be; without this, one cannot be precise about the details. If real space-times are a guide, one expects i to be singular, but the exact nature of the singularity is unclear. A second difficulty is that the structure of C_x is more complicated than in Minkowski space. In particular, in the examples studied thus far, it meets generic dual twistor lines in more than one point, and thus the googly maps are multi-valued. A further problem is the structure of $\mathscr{P}^*$. Since $\mathscr{P}^*$ is a family of subspaces of $\mathscr{M}$, its topology depends on how these lines

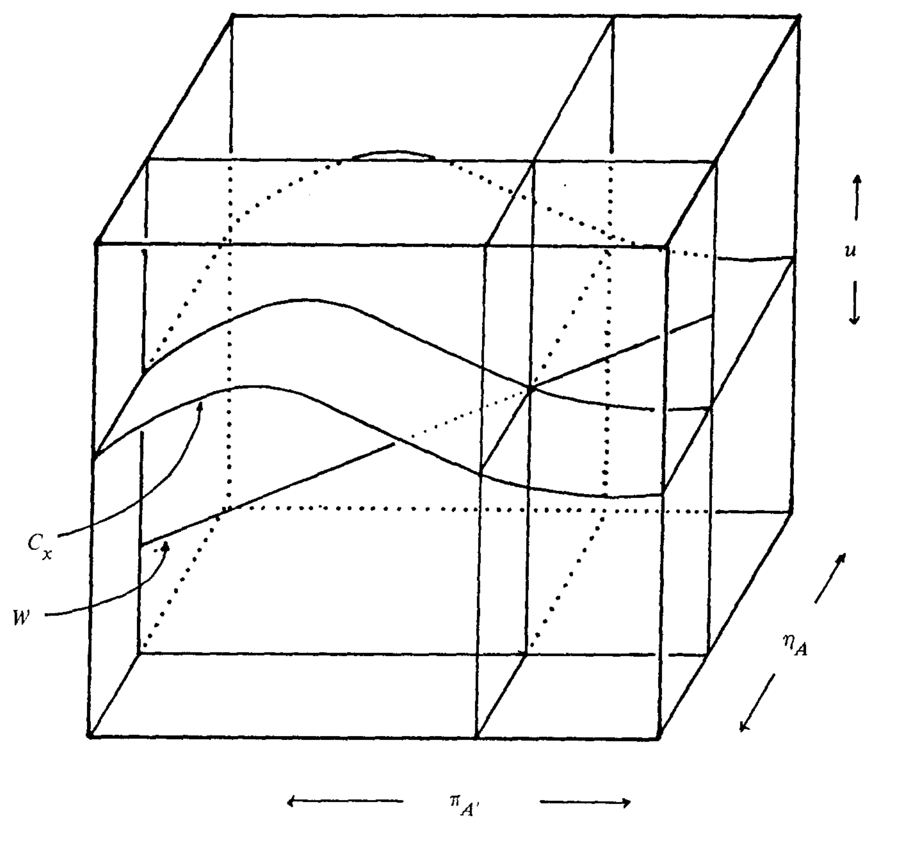


Figure 1. $\mathscr{I}$.

embed globally in $\mathscr{M}$. For a sufficiently small neighborhood 0 on $\mathscr{I}$, the structure induced on the set $\mathscr{P}(0)^*$ of dual twistor lines meeting 0 is canonically the same as that of a portion of flat dual twistor space $= \mathbb{C}P_3$. Globally, however, $\mathscr{P}^*$ can acquire a non-Hausdorff topology, and in fact this may be the generic case. Since the structure of $\mathscr{P}(0)^*$ is the same as for Minkowski space, it seems necessary to use that induced by all of $\mathscr{N}$ to encode the gravitational field. Again, in the absence of a clear understanding of the asymptotics of the space-time, elucidating this structure is difficult. Penrose has suggested blowing up a distinguished two-dimensional subspace (corresponding to the point i) in $\mathscr{T}^*$, the non-projective version of $\mathscr{P}^*$, as a way of resolving the singularity. This idea remains speculative, but receives some encouragement from the work described in this section.

An attractive feature of the Stein cover is that it can be used to define googly dual twistor space and googly maps without reference to the asymptotics of the space-time. Actually, we construct a family of dual twistor spaces, one for each $A \in \mathscr{P}$. Thus the total space G of googly dual twistors fibres over $\mathscr{P}$. G may be thought of as a «blown up» googly twistor space. If the asymptotics of the space-time are sufficiently nice, the point i corresponds to a line $I = L_i \subset \mathscr{P}$ and we will show $\mathscr{P}^* \subset G\,|_I$.

We begin with a nonlinear graviton (without points at infinity), and define the dual twistors relative to $A \in \mathscr{P}$. The key point is that U_A has naturally the structure of an open set in $\mathbb{C}P_3$. To see this, first note that A may be identified with a subset of $\mathbb{C}^2$ by the map

$$A \to P\Theta_A \text{ - vertical vectors} = \mathbb{C}P_2 - \mathbb{C}P_1 = \mathbb{C}^2$$

$$x \to \text{projective tangent to } L_x \text{ at } A$$

where Θ_A is the tangent space at A and «vertical» refers to the fibration $\mathscr{P} \to PS_{A'}$. Thus A has a natural affine structure, and it may be thought of as a subset of an α-surface in Minkowski space. The triviality of $P\mathscr{S}_{A'}$ over A allows this identification to be extended to the projective spin bundle. But then U_A may be identified with a subset of the twistor space of Minkowski space, which is $\mathbb{C}P_3$. To summarize, we have an embedding

$$U_A \to P_A \cong \mathbb{C}P_3. \tag{5}$$

We define

$$G_A = \{W \in P_A^* \mid \text{the plane } W \subset P_A \text{ meets the image of } U_A\},$$

the space of dual twistors relative to A. If $(\mathscr{M}, g_{ab})$ is a subspace of Minkowski space, there are canonical identifications among the various P_A's, and among the P_A^*'s.

The googly maps may now be defined. We first construct the covering space which is their domain. Fix $x \in \mathscr{M}$ and let

$$Q_A^* = \{(W, Z) \in G_A \times \mathscr{P} \mid Z \in W \cap L_x\}$$

Then the googly map is $\gamma_x : Q_A^* \to P\$_{A'}$ given by

$$\gamma_x(W, Z) = \pi(Z). \tag{6}$$

An explicit coordinatization for the points in U_A and G_A may be obtained as follows. Fix any point $0 \in A$, and a tangent spinor $\alpha_{A'}$. Then any point in A is of the form $\exp_0 \lambda^A \alpha^{A'}$ for some unique $\lambda^A \in \mathscr{S}_0^A$. The coordinates of a twistor $Z \in U_A$ relative to 0 are a pair of spinors $(\omega^A, \pi_{A'})$ where $\pi_{A'}$ is the tangent spinor to Z and $\omega^A = i\lambda^A \alpha^{A'} \pi_{A'}$ where $\exp_0 \lambda^A \alpha^{A'}$ is the point common to A and Z. (This coordinatization is chosen to agree with the usual twistor conventions. Note that $(\omega^A, \pi_{A'})$ is homogeneous of degree + 1 in $\pi_{A'}$; actually it specifies a non-projective twistor. The projective twistor is specified by the three independent ratios of the components). An element W of the dual projective space is then specified by a dual pair of spinors $(\rho_A, \mu^{A'})$, and defines the plane $\{(\omega^A, \pi_{A'}) \mid \rho \cdot \omega + \pi \cdot \mu = 0\}$. $Z = (\omega^A, \pi_{A'}) \in U_A$ and $W = (\rho_A, \mu^{A'}) \in G_A$ are *incident* if $\rho \cdot \omega + \pi \cdot \mu = 0$. In particular, although strictly $A \notin U_A$, A is a limit of points in U_A and we say $W = (\rho_A, \mu^{A'}) \in V_A$ is *incident to A* if $\alpha \cdot \mu = 0$. This is what one obtains by taking the limit of the previous definition as $Z \to A$.

Since A may be regarded as an α-surface in Minkowski space, the Minkowski-space incidence relations for twistors and dual twistors hold on A. In particular, the points on a null geodesic in A with tangent $\rho^A \alpha^{A'}$ define a set of lines in U_A which rule the plane $W = (\rho_A, 0) \subset U_A$. *There is thus a one-to-one correspondence between null geodesics in A and dual twistors both relative to and incident to A.*

We now consider the asymptotic dual twistor space. Suppose $(\mathscr{M}, g_{ab})$ has been completed with $\mathscr{N}$, and $\mathscr{P}$ is the twistor space of the completed manifold. As noted above, the assumption that i is regular is unrealistic. The arguments that follow can be altered to allow the sorts of asymptotic behavior generally considered in work on the googly. We shall not give the details here (they are easily supplied) in order to avoid a technical discussion of the asymptotics.

The point i corresponds to a line $I \subset \mathscr{P}$, the points of which may be identified with $P\mathscr{S}_i^A$. (In order for the assumptions of section 2 to be satisfied, it may be necessary to restrict $(\mathscr{M}, g_{ab})$ to a neighborhood of $\mathscr{N}$, or of $i \in \mathscr{M}$). Then all of the preceeding arguments apply, except the coordinatizations. In particular, a null geodesic lying in an η_A = constant surface on $\mathscr{I}$ is a dual twistor relative and incident to the twistor on I labelled by $\eta^A \in P_i^A$. Thus

PROPOSITION 5.1. *Dual asymptotic twistor space may be identified as the subset* $\mathscr{P}^* = \{(A, W) \in G|_I \mid A \text{ is incident to } W\}$. ∎

The space $G|_I$ may be thought of as similar in spirit (and nearly identical) to Penrose's blown up $\mathbb{P}^*$. It is straightforward to check that the definition (6) of the googly maps, restricted to $\mathscr{P}^*$, agrees with the previous one.

It should be noted that a similar googly space can be defined by replacing i with any other point. The only reason for choosing i is that it is distinguished.

Although the introduction of G does not lead to an obvious solution of the googly problem, there are some encouraging signs. First, there is a non-singular definition of the googly maps. Second, the projective tangent bundle to $\mathscr{P}$, which may be thought of as a «linearization» of G, has featured in some proposed attacks on the googly [11, 17]. There is also a close link with Penrose's recent work on googlies and asymptotic local twistor transport [19]. The map (5) which embeds U_A in flat twistor space can also be defined by local twistor transport around A, and similarly G_A is a subset of the space of sections of the bundle of dual local twistors over A which are covariantly constant under local twistor transport. Third, G can be used with some success to give a googly description of massless fields (a googly Penrose transform). The success is so far less than total because no way has been found to remove reference to $\mathscr{P}$.

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